JOURNAL OF APPROXIMATION THEORY 38, 60–65 (1983)

Extremal Points for Some Sets of Projections on $C(I)$

WILLIAM A. LIGHT

Department of Mathematics,
University of Lancaster, Lancaster, LA1 4YL, England

Communicated by Oved Shisha

Received September 30, 1981; revised March 17, 1982

1. INTRODUCTION

We shall consider the continuous functions on a closed interval I. The Banach space $C(I)$ with the supremum norm $\|x\| = \sup_{t \in I} |x(t)|$ will be denoted by x. We suppose that Y is an n-dimensional Haar subspace of x containing constant functions, L is a projection of X onto Y if $L \in B[X, Y]$, and L is idempotent. Given a class F of projections of x onto Y, L^* is termed minimal in F if $\inf_{L \in F} \|L\| = \|L^*\|$, where $\|L\| = \sup_{\|x\|=1} \|Lx\|$. If the class F contains *all* projections of x onto Y, then there exists such a minimal projection, but it has no known characterizations.

The only nontrivial class in which the minimal projection may be characterized is the class of *interpolating projections*. These projections may be written in the form

$$L = \sum_{i=1}^{n} \hat{t}_i \circ p_i, \tag{1}$$

where the $\hat{t}_i$ are point evaluation functionals corresponding to the n distinct interpolation points in I, and the p_i are the familiar Lagrange interpolation polynomials satisfying $p_i(t_j) = \delta_{ij}$, where δ_{ij} is the Kronecker delta. The minimal projection in this class is characterized by referring to the function $g_L(t) = \|\hat{t} \circ L\|_{x^*} = \sup_{\|x\|=1} |(Lx)(t)|$. If the endpoints of I are included in the set of interpolation points, then the minimal interpolating project is such that $\|\hat{t} \circ L\|_{x^*}$ has $n-1$ equal extrema; see [4] for details.

In this paper we shall explore some of the structure of the set $A_k = \{L: L \in B[X, Y],\ Ly = y\ \forall\, y \in Y,\ \|L\| \leqslant k\}$. In particular we shall identify exactly which of the interpolating projections in A_k are extremal points of this set. Of course, A_k might be the empty set, since $\|L\|$ is bounded below by the norm of the minimal projection L^*. Even if A_k is nonempty, it may not contain any interpolating projections since the minimal projection is *not* in general an interpolating projection.

0021-9045/83 $3.00

The existence of extremal points for the set A_k is not immediately obvious, but can be deduced either by arguments similar to those contained in [1] or by the following reasoning: any bounded projection $L \in A_k$ can be represented in the form

$$L = \sum_{i=0}^{n} f_i \otimes p_i,$$

where the $f_0, f_1, \ldots, f_n$ are independent bounded linear functionals on x, and $p_0, p_1, \ldots, p_n$ are members of Y determined by $f_i(p_j) = \delta_{ij}$, where δ_{ij} is the usual Kronecker delta. If we normalize the p_i to have unit norm and denote the Cartesian product of X^* with itself $n+1$ times by $(X^*)^{n+1}$, then to each projection $L \in A_k$ there corresponds a unique $\mathbf{f} \in (X^*)^{n+1}$, where $\mathbf{f} = (f_0, f_1, \ldots, f_n)$. We shall need some results about the w^*-compactness of certain of these sets. It should be noted that the following result does not depend on the spaces X and Y being identified with $C(I)$ and an n-dimensional subspace of $C(I)$. Any Banach space X and finite-dimensional subspace Y will suffice.

LEMMA 1. *Let C be the set of all $\mathbf{f} \in (x^*)^{n+1}$ such that $f_i(p_j) = \delta_{ij}$ and $L = \sum_{i=0}^{n} f_i \otimes p_i$ is a member of A_k. Then C is a w^*-compact subset of $(x^*)^{n+1}$.*

Proof. For $1 \leqslant i, j \leqslant n$, define the maps $u_{i,j}$: $(x^*)^{n+1} \to \mathbb{R}$ by $u_{i,j}(\mathbf{f}) = f_i(p_j)$. Then it is immediate from the definitions that $u_{i,j}$ is continuous when the w^*-topology is imposed on $(X^*)^{n+1}$ and the usual topology on $\mathbb{R}$. Now consider also the maps $v_{x,t}$: $(X^*)^{n+1} \to \mathbb{R}$ given by

$$v_{x,t}(\mathbf{f}) = \left| \sum_{i=0}^{n} f_i(x) p_i(t) \right|.$$

Again these are continuous mappings from $(X^*)^{n+1}$ into $\mathbb{R}$. Now setting

$$D = \bigcap_{i,j} \{\mathbf{f}: u_{i,j}(\mathbf{f}) = \delta_{ij}\}, \qquad E = \bigcap_{\substack{t \in I \\ \|x\| \leqslant 1}} \{\mathbf{f}: v_{x,t}(\mathbf{f}) \leqslant k\},$$

then D and E are w^*-closed, and since C is the intersection of D and E, C is also w^*-closed. It is clearly bounded, and so is w^*-compact.

COROLLARY. *The set A_k is the closed convex hull of its extreme points.*

We shall now stipulate that Y is the subspace of polynomials of degree $n-1$, where $n \geqslant 3$. This has the consequence (see [3]) that all projections from X to Y have norm strictly greater than unity. We shall denote the closed unit sphere in X by $S(X)$ and the set $\{x: x \in X, Lx = 0, L \in B[X, Y]\}$ by $\operatorname{Ker}(L)$.

2. The Functions g_L

The function g_L is often referred to as the Lebesgue function of the projection L. If L is an interpolating projection, then g_L has several special properties. Henceforward we assume Y is the subspace of polynomials of degree at most $n-1$ when the following may be found in [6].

Property 2–1. We have that

$$g_L(t) = \|\hat{t} \circ L\|_{X^*} = \sum_{i=1}^{n} |p_i(t)|.$$

Property 2–2. The function g_L is a piecewise polynomial of degree $n-1$ with knots at $t_1, t_2, ..., t_n$, the t_i being the interpolation points of L.

Property 2–3. Let $I = [a, b]$ and the interpolation points be ordered so that $t_1 < t_2 < \cdots < t_n$. Then g_L is strictly increasing and convex in $[t_n, b]$ and strictly decreasing and convex in $[a, t_1]$.

Property 2–4. The function $g_L(t)$ has exactly one maximum value in each of the intervals $[t_i, t_{i+1}]$ for $1 \leqslant i \leqslant n-1$, and at these points $g_L''(t) < 0$.

In fact, Property 2–4 differs slightly from [6, Property A–5], but Property 2–4 is established in the course of the proof of Property A–5.

Lemma 2. *Let $L \in A_k$ be an interpolating projection with $\|\hat{s} \circ L\|_{X^*} = k$ for some $s \in I$. Suppose $H, K \in A_k$ and $L = \theta H + (1-0) K$ for $0 < \theta < 1$. Then if $x \in \operatorname{Ker}(L)$ we have $(Hx)(s) = (Kx)(s) = 0$. Furthermore, if s is an interior point of I, we have that the derivates $(Hx)'(s)$ and $(Kx)'(s)$ are also zero.*

Proof. The kernel of L consists of functions $x \in X$ such that $x(t_i) = 0$, $1 \leqslant i \leqslant n$, where the t_i are the interpolation points of L. Now define

$$Z = \{z \in X: \|z\| = 1 \text{ and } z(t_i) = \operatorname{sgn} p_i(s), 1 \leqslant i \leqslant n\},$$

where $(Lx)(t) = \sum_{i=1}^{n} x(t_i) p_i(t)$. It is clear that $(Lz)(s) = k$ for all $z \in Z$. Furthermore, since $H, K \in A_k$ and $\|L\| = k$, we must have $\|H\| = \|K\| = k$, and similarly $(Hz)(s) = (Kz)(s) = k$ for all $z \in Z$. Now if s is an interior point of L, then the derivatives $(Hz)'(s)$ and $(Kz)'(s)$ must be zero.

Now pick $x \in \ker(L)$ such that $\|x\| \leqslant 1$. Then since $\|\hat{t} \circ L\|_{X^*}$ is continuous and greater than unity for $t \in (t_i, t_{i+1})$ and $0 \leqslant i \leqslant n+1$, where t_0, t_{n+1} are the end points of the interval I, we can construct disjoint neighbourhoods N_i of the t_i, $1 \leqslant i \leqslant n$, such that $|x(t)| < \delta < 1$ for all $t \in N_i$. Now take M_i as neighbourhoods of the t_i such that $\bar{M}_i \subseteq N_i$. Then choose $z \in C(I)$ satisfying $z(t) = 0$ for $t \notin \bigcup_{i=1}^n N_i$, $z(t) = \{1 - |x(t)|\}$ sgn $p_i(s)$ for $t \in M_i$ and $\|z\| \leqslant 1 - \delta$. Then clearly $z \in Z$ and we claim $x + z \in Z$. Clearly, we need only establish $\|x + z\| \leqslant 1$.

For $t \in N_i$ we have

$$|x(t) + z(t)| = |\{1 - 1x(t)|\} \operatorname{sgn} p_i(s) + x(t)|$$
$$\leqslant 1 - \delta + \delta = 1.$$

For $t \notin \bigcup N_i$ we have

$$|x(t) + z(t)| = |x(t)| \leqslant 1.$$

It now follows immediately that $(Hx)(s) = (Kx)(s) = 0$ and $(Hx)'(s) = (Kx)'(s) = 0$ if s is an interior point of I.

4. Proof of the Main Theorem

We shall in this section assume that k is sufficiently large for A_k to contain interpolating projections. To simplify the statement of the theorem, we shall introduce the notion of a k-maximum. The function $x \in X$ will be said to have a k-maximum at $t = s$ if x has a local maximum there, and $x(s) = k$.

THEOREM. *Let $L \in A_k$ be an interpolating projection with interpolation points $t_1, t_2, \ldots, t_n$. Then L is an extremal point of A_k if and only if the following conditions hold:*

(i) *If* dim Y *is odd, either $\|\hat{t} \circ L\|_{X^*}$ has at least $\frac{1}{2}(n+1)$ k-maxima in (t_1, t_n) or at least $\frac{1}{2}(n-1)$ k-maxima in (t_1, t_n) and $\|\hat{t} \circ L\|_{X^*} = k$ at one of the endpoints of I;*

(ii) *If* dim Y *is even, either $\|\hat{t} \circ L\|_{X^*}$ has at least $[\frac{1}{2}(n+1)]$ k-maxima in (t_1, t_n) or at least $[\frac{1}{2}(n-1)]$ k-maxima in (t_1, t_n) and $\|\hat{t} \circ L\|_{X^*} = k$ at both the endpoints of I.*

Proof. We begin by establishing the sufficiency part of the theorem. The proof rests on the fact that if $y_1, y_2 \in Y$ are equal and have their derivatives equal at enough points, then $y_1 \equiv y_2$. We shall use this property to show that if $L = \theta H + (1 - \theta) K$, then $\ker(L) \subset \operatorname{Ker} H$ and $\ker(L) \subset \ker(K)$; since L, H, K are projections onto the same subspace $L \equiv H \equiv K$. We give as an example the proof for dim Y even, and $\|\hat{t} \circ L\|_{X^*}$ having at least $[(n-1)/2]$

k-maxima in (t_1, t_n) and $\|\hat{t} \circ L\|_{X^*} = k$ at each of the endpoints of I; all the other cases follow similarly.

Suppose L can be written as $L = \theta H + (1 - \theta) K$, where $H, K \in A_k$ and $0 < \theta < 1$. Let $x \in \ker(L)$. There are at least $[\frac{1}{2}(n-1)] + 2$ points at which $\|\hat{t} \circ L\| = k$ and by Lemma 2 Hx and Kx are zero at these points. Also at $[\frac{1}{2}(n-1)]$ of these the derivative of $\|\hat{t} \circ L\|$ exists and is zero by Property 2–4. An application of Lemma 2 shows that $(Hx)'$ and $(Kx)'$ vanish at these points. These $[\frac{1}{2}(n-1)] + [\frac{1}{2}(n-1)] + 2 = n$ conditions are sufficient for $Hx \equiv Kx \equiv 0$, which, in turn, implies $\ker(L) \subset \ker(H)$ and $\ker(L) \subset \ker(K)$, completing the proof of sufficiency.

We shall now establish the necessity in the case dim Y is odd, since the corresponding proof for dim Y even is very similar. Suppose first that $\|\hat{t} \circ L\|$ has at most $\frac{1}{2}(n-1)$ k-maxima in (t_1, t_n), and $\|\hat{t} \circ L\| < k$ at each endpoint of I. Now construct $y_0 \in Y$ such that y_0 has zeros and does not change sign at each of the points at which $\|\hat{t} \circ L\|$ has k-maxima. Take a functional $\phi \in X^*$ with the property $\phi | Y \equiv 0$, and define an operator $R \in B[X, Y]$ by $Rx = \phi(x) y_0$. Then $R | Y \equiv 0$ and Rx has zeros and does not change sign at each of the k-maxima of $\|\hat{t} \circ L\|_{X^*}$, for any $x \in X$. Now for any $\theta \in \mathbb{R}$, $L + \theta R$ is a projection from X onto Y. Let the k-maxima of L occur at $s_1, s_2, \ldots, s_r$, where $r \leqslant \frac{1}{2}(n-1)$. Surround these points by open intervals $N_1, N_2, \ldots, N_r$ such that $\|\hat{t} \circ L\|_{X^*} > a > 1$ for all $t \in N_i$, $1 \leqslant i \leqslant r$. By Property 2–4 these intervals will be disjoint. Also by this property, $\|\hat{t} \circ L\|_{X^*}$ has nonzero second derivative in each of the N_i and so there exists $\delta_1 > 0$ such that

$$\|\hat{t} \circ (L + \theta R)\| \leqslant k \qquad \text{for all} \quad t \in \bigcup_{i=1}^{r} N_i \quad \text{and} \quad |\theta| < \delta_1.$$

Furthermore $I \backslash \bigcup_{i=1}^{r} N_i$ is closed and $\|\hat{t} \circ L\|_{X^*}$ is a continuous function of t on this set, and so $\|\hat{t} \circ L\| \leqslant M < k$ for $t \in I \backslash \bigcup_{i=1}^{r} N_i$. Consequently, there exists a $\delta_2 > 0$ such that

$$M + |\theta| \, \|\hat{t} \circ R\| \leqslant k$$

for

$$t \in I \backslash \bigcup_{i=1}^{r} N_i \qquad \text{and} \qquad |\theta| < \delta_2.$$

Now for $|\theta| < \min\{\delta_1, \delta_2\}$ we have $\|\hat{t} \circ (L + \theta R)\| \leqslant k$ for all $t \in I$. Let θ_0 be such a value; then $H = L + \theta_0 R$ and $K = L - \theta_0 R$ are both in A_k and we can write $L = \frac{1}{2} H + \frac{1}{2} K$, which shows that L is not an extremal point of A_k.

The only case remaining for which the dimension of Y is odd occurs when $\|\hat{t} \circ L\|_{X^*}$ has at most $\frac{1}{2}(n-3)$ k-maxima and $\|\hat{t} \circ L\|_{X^*} = k$ at one or both of the endpoints. In this case we construct the y_0 in the operator R such that y_0 has zeros and does not change sign at each of the k-maxima of $\|\hat{t} \circ L\|_{X^*}$. We also require that y_0 has a zero at the end-points if $\|\hat{t} \circ L\| = k$ at these points. A similar argument to the one above, but invoking Properties 2–3 and 2–4, yields the existence of H, K such that $L = \frac{1}{2}H + \frac{1}{2}K$. As was remarked earlier, the proof when Y is of even dimension goes through in an analogous manner.

5. Remarks

Despite the fact that we have given here necessary and sufficient conditions for interpolating projections to be extremal points, these clearly cannot constitute all the extremal points of A_K, since for sufficiently small k, A_k is known not to contain any interpolating projections. In view of this, it would be interesting to know what the other extremal points look like. For example, a necessary condition for a projection to be an extremal point of A_K would provide us with some information about the minimal projection. The work of Cheney *et al.* [2] involves arguments of this nature.

Acknowledgment

The author would like to thank the referee for his comments on an earlier version of this manuscript.

References

1. J. Blatter and E. W. Cheney, On the existence of extremal projections, *J. Approx. Theory* **6** (1972), 72–79.
2. E. W. Cheney, C. R. Hobby, P. D. Morris, F. Schurer, and D. E. Wulbert, On the minimal property of the Fourier projection, *Trans. Amer. Math. Soc.* **143** (1969), 249–258.
3. E. W. Cheney and P. D. Morris, On the existence and characterization of minimal projections, *J. Reine Angew. Math.* **270** (1974), 61–76.
4. C. de Boor and A. Pinkus, Proof of the conjectures of Bernstein and Erdös concerning the optimal nodes for polynomial interpolation, *J. Approx. Theory* **24** (1978), 289–303.
5. N. Dunford and J. Schwartz, "Linear Operators, Part I," Interscience, New York, 1958.
6. F. W. Luttmann and T. J. Rivlin, Some numerical experiments in the theory of polynomial interpolation, *IBM J. Res. Develop.* **9** (1965), 187–191.